W. E
Sid. Coll

# A
# REPLY
# TO A
# PAMPHLET
# ENTITLED
# OBSERVATIONS
# ON THE
# FIRST CHAPTER of a BOOK
# CALLED
# MISCELLANEA ANALYTICA.

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BY
**EDWARD WARING, M.A.**

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*CAMBRIDGE,*
Printed by J. BENTHAM, Printer to the UNIVERSITY;
for W. THURLBOURN & J. WOODYER, in *Cambridge*.

M.DCC.LX.

BRITISH MUSEUM

# A REPLY, &c.

THE *Obſerver's* Objections to my Book are chiefly general; and every one knows how difficult it is to anſwer general Objections.

He ſays my Lemmas are Problems, and my Corollaries are Problems. But why a Lemma or a Corollary may not be a Problem, I am ſure I don't know: certainly they are often ſo in the beſt Books that are extant.

He is very ſevere and cruel in his Reflections upon the Obſcurity and Perplexity of my Manner of Writing. I hope I am not ſo obſcure as he repreſents me to be. The Reader muſt be the Judge. I only deſire him to remember, that I am a young Man, and that it is my firſt Work, and principally, that my View in publiſhing it, was not to ſhew how well I can write; but to ſhew, that I have ſpent a great deal of Time and Labour in the Study of this Science.

He quotes a long Paſſage out of my Book to prove how bad a Writer I am: but ſurely I am not ſo bad a Writer, as he is an unkind and unfair Interpreter: no Reader, that was impartial, could imagine, that I uſed *tentando* in the Senſe that He interprets it. He aſks if the ſecond Clauſe refers to Queſtions or Demonſtrations.

My

My Words are, *Quæſtiones in genere vel ſunt Regulæ, quæ*, &c. *vel conſtant*, &c. where *Quæſtiones* is ſo clearly the Nominative Caſe to *conſtant*, that it is impoſſible to miſtake it. The next Words, that he finds Fault with, are *Quantitates dati Generis, & haud dati Generis Quantitates:* — which He tranſlates, *Quantities of a given Kind, and of a Kind not given:* — as if the Words referred to *conſtant* and *variable* Quantities, though He acknowledges they do not. The *Genera* here referred to, are *affirmative*, *negative*, and *impoſſible* (ſo all Writers call them), which are as well known, as *determined* and *variable*; though he ſpeaks of theſe as the only *Genera* that have been ever heard of.

To defend ſingle Expreſſions, or to anſwer general Accuſations, is a diſagreeable Taſk: I will leave theſe therefore to the Good Nature of my Reader, and go on to the *Obſerver's* particular Objections.

But I muſt firſt complain of the Manner in which theſe Obſervations are written; which is extreamly cruel from one End to the other of them. The very Title Page is unfair: He calls them, *Obſervations upon the Firſt Chapter:* by which He would perſuade his Readers to believe, that He has found Faults enough, even in my *firſt* Chapter, to fill a Pamphlet; whereas in Fact he goes through my whole Book. And I have the more Reaſon to complain of this, as the *ſingle* Error that He has found, is not in the *firſt* Chapter but in the *laſt*.

He there obſerves, that, in dividing one Quantity by another, I have a (+), in the Quotient, inſtead of a (—). This is true; and I am perſuaded, that the Reader will excuſe ſo ſmall an Error; eſpecially if he finds, as I hope he will, that of all the Objections, which the *Obſerver* makes to my Book, *this is the only one, in which there is even the Appearance of Truth.*

He begins with my firſt Lemma, and I wiſh he had made his Objections more particular: He complains "that after having conducted my Readers thro' one very "eaſy Step, I leave them to themſelves to "ſolve this difficult Problem."

The *Obſerver* calls what I have ſaid on this Subject a *Solution*; he ſhould have call'd it a *Demonſtration*: for it is a *Demonſtration*, and not a *Solution*.

He ſays there is but one Step in my Demonſtration; — there are five, and I know not how to put in more.

He acknowledges in the next Page, that Dr. *Saunderſon* has prov'd the Truth of my Lemma in Biquadratic Equations. — Now my Reaſoning, upon the general Caſe, is juſt the ſame as his, in the ſpecial Caſe. If *Saunderſon* therefore be intelligible, I muſt be ſo too; and if *Saunderſon's* be a Proof, mine muſt be ſo too. Indeed, tho the *Obſerver* complains of my Brevity, the Reader, who ſhall take the Trouble to make the Compariſon, will find, that *Saunderſon* is ſhorter than I am. But, towards the Bottom of the Page, he ſeems to have chang'd his Opinion: for he ſays, " that *Saunderſon*'s " Proof does not ſhew the Neceſſity of uſing " an Equation of ſix Dimenſions; ſince a Cubic, whoſe three Roots may have each two " Values, when the Signs ONLY are changed, " will ſerve the Purpoſe." (*Obſervat.* p. 6, 7.) — As what the *Obſerver* ſays here is not very intelligible, I ſhall endeavour to explain it. *Des Cartes* in order to reduce a Biquadratic Equation to a Quadratic, makes uſe of an Equation of ſix Dimenſions of this Form;

$$x^6 - px^4 + qx^2 - r = 0.$$

Suppoſe $\left\{\begin{array}{l} x^2 = z \quad \text{and you will have} \\ z^3 - pz^2 + qz - r = 0 \end{array}\right.$: and this is the Cubic Equation, which, the *Obſerver* ſays, is ſufficient for the Reduction of the Biquadratic. But the *Obſerver* is miſtaken: for the Biquadratic is not reduced by means of this Cubic alone, but by means of the

$$\begin{array}{r} \text{Cubic} \\ \text{and Quadratic} \end{array} \left\{\begin{array}{l} z^3 - pz^2 + qz - r = 0 \\ x^2 = z, \quad \text{both together.} \end{array}\right.$$

The Fact is this: every Equation of ſix Dimenſions, which has only one unknown Quantity, may, by the Introduction of another unknown Quantity, be reſolved into a Cubic and Quadratic Equation. And on the other Hand, if you have a Cubic and a Quadratic, with two unknown Quantities, and would exterminate one of them; this will bring you back again to an Equation of ſix Dimenſions. To reduce the Biquadratic therefore, by one Equation of ſix Dimenſions, and one unknown Quantity; or two Equations, (one a Cubic, and the other a Quadratic,) with two unknown Quantities; this is exactly the ſame Proceſs.

But the *Obſerver* thinks, that the Cubic Equation alone is ſufficient for the Purpoſe; becauſe its three Roots may have each *two* Values, when the Signs *only* are changed. ——To underſtand this the Reader muſt be told, that *Saunderſon* has proved, that the Equation by which the Biquadratic is reduced muſt have ſix Roots; which the *Obſerver* believed: and believing too, that the Cubic alone was ſufficient for the Reduction, it behoved him to find in it ſix Roots: and he finds them by *only* changing the Signs; as if indeed the Signs of an Equation could be thus changed, and the Equation ſtill continue the ſame.

The Roots of the Cubic Equation $x^3 - 2x^2 - 5x + 6$, are $+1, -2, +3$: Change the Signs of the 2d and 4th Terms, and you'll have, $x^3 + 2x^2 - 5x - 6 = 0$; of which, $-1, +2, -3$, are the Roots. And this is what the *Obſerver* means, when he ſays, that *The Roots of a Cubic Equation may each have two Values, when the Signs*

ONLY *are changed.* And does he really think, that we can thus change the Signs, without changing the Equation? If he does, he muſt think too, that a Cubic Equation may have ſix Roots; which indeed was what he wanted to make out, tho' the Impoſſibility of its having more than three, is demonſtrated in almoſt every Book of Algebra that is extant.

What the *Obſerver* objects further to the Lemma, is ſo evidently not to the Purpoſe, that one can hardly believe him ſerious: He ſays, "the firſt and moſt obvious way of reducing an Equation is, "by the Help of the ſimple Equations out "of which it is formed" (p.7.). This is certainly true; but the Equation muſt be firſt ſolved, before this Method can be put in Practice: and my Lemma is not to ſhew, how Equations may be reduc'd after they are ſolved; but how to reduce them, in order to their Solution. He does me ſo much Honour, as to acknowledge, that I have reduc'd Equations of 4, 5, and 6 Dimenſions. He adds however, that theſe very Reductions prove my Rule to be falſe in theſe particular Caſes. It would be very hard indeed if they did; ſince it was by the Help of this very Lemma, that I made ſome of thoſe Reductions.

But to conclude this Head: the *Obſerver* could not but underſtand, that my Lemma relates to a general Method: the Particulars therefore which he ſpecifies, are beſide the Purpoſe; becauſe for particular Caſes, peculiar Facilities may be invented, which will not extend to Equations in general. I proceed to the ſecond Lemma.

" The ſecond Lemma (ſays the *Obſerver*) " propoſes to find the Sum of the Roots of a " given Equation, the Sum of the Squares " of each Root, of the Cubes, or of any " propos'd Powers. For the Solution of this " Problem, Sir *Iſaac Newton* has given us a " Rule, which is clear, ſhort, neat, eaſy to " be underſtood, remember'd, and applied. " This Rule is here repreſented in a new " Form; and has loſt its original Simplicity " and Conciſeneſs. But this is not all which " it has ſuffer'd. For, tho' it was not difficult, " as every one acquainted with theſe Subjects " will perceive, to change its Form, yet this " is not done here without a conſiderable " Miſtake". (p.8).—Eaſy as the *Obſerver* takes this to be, it has exercis'd the beſt Heads of the laſt Century. I ſay this, to ſhew that I have not employed my Time and Labour upon a contemptible Subject, as the *Obſerver* would make his Readers believe. This Problem, of which he ſpeaks with ſo much Scorn, is in *De Moivre's Miſcellanea Analytica.* That excellent Analyſt, as the *Obſerver* deſervedly calls him, has many Pages upon the Subject. Even the immortal Pen of *Newton* has been employ'd upon the ſame Problem; only *changing the Expreſſions,*

*preſſions*, (as the *Obſerver* thinks,) and deſtroying *the original Simplicity and Conciſeneſs* of the excellent Rules, which he had receiv'd from his Maſters.

What he ſays of Sir *Iſaac Newton's* Rule, *viz.* that it is clear, *&c.* is certainly true: and if I have done Nothing but change the Expreſſion of this Rule, I have ill-beſtow'd my Time and Labour indeed. But the Truth is, that I have ſolv'd another Problem. Sir *Iſaac Newton's* Rule teaches us, how to find the Sum of the Roots of any Equation, the Sum of their Squares, of their Cubes, *&c.* one after another. My Problem is, to find the Sum of any propos'd Power of the Roots *per Saltum*, without firſt finding the Sum of the inferior Powers.

Before the Invention of the Binomial Theorem, every Body knew, how to find the ſeveral Powers of a Binomial, one after another, with extream Facility: and yet the Binomial Theorem, which teaches us, how to find any propos'd Power *per Saltum*, without firſt finding the inferior Powers, is admir'd by all the World.

I hope my Reader will not ſuſpect me of Vanity, as if I compar'd myſelf with theſe very great Men, or my Problem with any thing they have done. The *Obſerver* has forc'd me to ſay all this, in order to ſhew, that my Problem is not, in itſelf, contemptible. I proceed therefore, to conſider the Objections he has made to my Solution of it; in which, he thinks, I have made a conſiderable Miſtake.

“ *N*, the Index of the higheſt Term of the “ Equation, is ſuppoſed alſo to be the In- “ dex of the Powers, whoſe Sum is to be “ found. And thus the Solution is made far “ leſs general than the Problem. Inſtead of “ teaching us, how to find the Sum of the “ Roots, of their Squares, of their Cubes, “ &c. in any Equation; it ſhews us only, “ what is the Sum of the Roots in a ſimple “ Equation, of their Squares in a Quadratic, “ of their Cubes in a Cubic, &c.”

As to what the *Obſerver* ſays, that my Rule ſhews *only*, how to find the *Sum* of the *Roots* in a *ſimple* Equation; I muſt obſerve, what every Body knows, that a *ſimple* Equation can have only *one* Root.

Then as to the *N*, which I have made the Index of the higheſt Term of my Equation; I am very ſorry, that I did not uſe ſome other Letter; as this, it ſeems, would have prevented the *Obſerver's* Miſtake: for indeed, it is a grievous Miſtake, to imagine, that this makes my Solution leſs general, than it would otherwiſe have been.

Though the *Obſerver* would not give himſelf the Trouble to find out the Analyſis of my Problem, nor even to underſtand my Demonſtration of it; yet Sir *Iſaac Newton's* Rule, which he had before his Eyes, if he had at all attended to it, might have ſaved him from falling into this Error. He might have obſerved, that the Index of the higheſt Term of the Equation, never comes into Conſideration; that there is no Notice taken of it, either in the Rule itſelf, or in the Reaſoning, from whence the Rule is derived. The Reaſoning is the ſame; the Rule is the ſame; the very Terms of the Series are the ſame, for Equations of all Degrees; with this ſingle Difference, that the higher the Equation, the more Terms the Series may have in it. It is indeed cuſtomary, to write down an Equation; but even to write down an Equation is meer matter of Form. If I had ſaid let $p, q, r, s$, &c. in their Order, be the Coefficients of any Equation, this had been fully ſufficient: nay, the very Mention of an Equation is not at all neceſſary. For the Problem is ſimply this.

Having given ($p$) the Sum of any Number of Quantities, ($q$) the ſum of their Products, when taken two and two; ($r$) the Sum of their Products, taken three and three; &c. to find the Sum of their Squares, Cubes, Biquadrates, &c.

In a caſe therefore, where the very Mention of an Equation was not at all neceſſary; if, like others, I have, for Form ſake, put an Equation at the Head of my Solution; whatever may be the Index of the higheſt Term of my Equation, it plainly can be a Matter of no Importance at all.

But the *Obſerver* goes on, under the falſe Perſuaſion of my Solution's not being general, and adds, "It ſeems indeed that our "Author afterwards diſcovered, that he "had anſwered only a ſmall part of the "Queſtion. But inſtead of correcting the "Miſtake, which limited his Solution; he "gives us in the third Page, a number of "Rules, by which we may adapt it to each "particular Caſe." Surely the *Obſerver* never read this third Page; certainly He never read it with any Attention: for this third Page contains a full *ſynthetical Demonſtration* of the Lemma. It is indeed a peculiar Species of Demonſtration; it was firſt invented by *James Bernoulli*: he was a ſevere Mathematician, and never contented, ſo long as he had left any Doubt upon the Mind of his Reader. He invented therefore this Species of Demonſtration, for the Proof of ſuch general Propoſitions, as are derived, by Induction, from particular Caſes. The Binomial Theorem is of this Nature: and I have ſeen it thus proved in Books of common Algebra. Suppoſe $\overline{1+x}$ to be the Binomial. The peculiarity of this Species of Demonſtration is, that it does not prove directly, that the Theorem is true; but ſuppoſing the Theorem to be true, in the $(n^{\text{th}})$ Power, for Example; it proves, that it will be true likewiſe in the $(\overline{n+1}^{\text{th}})$ Power.

You may by Multiplication find the Square, or Cube, or higher Powers, if you pleaſe; but this is not at all neceſſary: for by this Method it is proved, that if the Theorem be true for the Square, when $n=2$; then it will be true in the Cube alſo, when $n+1=3$; and if in the Cube, then in the 4th Power, and ſo on: which makes the Proof univerſal. And this then is the general Nature of this Species of Demonſtration; to ſhew, that if the Propoſition be true in one, two, or more Caſes, according to the Circumſtances of the Propoſition, then it will be true in all Caſes whatever. Now having derived my Solution of this Problem, by Induction, from a Table of the lower Caſes, calculated by Sir *Iſaac Newton's* Rule; this Species of Demonſtration ſeemed peculiarly adapted to it. It ſhould ſeem, that the *Obſerver* has not been accuſtomed to this kind of Demonſtration, though it be of great Extent: and being under a falſe Perſuaſion that my Solution was not general, He miſtook my Demonſtration for a Series of particular Rules; which no unprejudiced Perſon, to whom this Kind of Proof was familiar, could have done.

" Such a Series as this, is of no Uſe, unleſs we underſtand by what Law it is form'd: the writer undertakes to teach us; but he does it both obſcurely and imperfectly".— All that I can ſay in Anſwer to this, is, that I have explain'd the Law of this Series with ſo much Perſpicuity, that I am very ſure no Mathematician can doubt about my meaning, or err in the Formation of the Series. — But, ſuppoſing my Words were liable to be miſunderſtood, yet the Symbols are not to be miſtaken. I have calculated the Series to ſeven Terms; enough to ſhew the Law of it to a Reader of any Practice and Sagacity, tho' I had not ſaid a word about it. —— But my words are not ambiguous. The *Obſerver* owns they are the very words which *De Moivre* uſes on another Occaſion. But he adds, that *De Moivre* explain'd the words; and he is very angry at me for not having done ſo too. Now to me it ſeem'd, that *De Moivre*'s having explain'd the words, made my Explication of them unneceſſary. If every technical Term was to be explained in every Book that was written, there would be no End of the Books that ſhould be written. But that is not all. *De Moivre* introduced theſe Terms in the Explication of his Multinomial Theorem. No Mathematician but has read it. The Terms in conſequence of this are become notorious; and to have explained them again, would not only have been unneceſſary; but moſt Readers would have thought it impertinent.

But the *Obſerver* tells us, that *De Moivre* has given us another Clauſe to ſupply the Defect of his Rule; and that I ought to have put in the ſame Clauſe, or *ſomething* *like it.* Upon which, I hope, I may be permitted to take notice, that if the *Obſerver* had any where found, in my Book, an Expreſſion of ſo little Preciſion as this *ſomething like it*; he would not have failed to have animadverted upon it with great Severity. And all the Anſwer that I can make to his Aſſertion is, what I have ſaid before, that no Reader, of any Experience in theſe Matters, can poſſibly fall into a Miſtake.

## LEMMA II. COR. I.

Pag. 10. "It may, perhaps, puzzle an attentive Reader, to diſcern from the Words "in which they are propoſed, the Difference "between the firſt Corollary to this Propoſition, and the Propoſition itſelf. But, if the "Series, which follows be examined, it will "appear, that the Deſign of the Corollary "is not to find the Sum of the Roots, or the "Sum of their Squares, or the Sum of any "other Powers, but the Sum of their Sums." Whether the Solution of my Problem was difficult to him or eaſy; it ſurely was not difficult for the *Obſerver* to have formed a Table of the lower Caſes from Sir *Iſaac Newton's* Rule. And if he had done this, which any candid Examiner would have done; He muſt, I think, have ſeen at the firſt Glance of his Eye, not only the meaning of my Corollary, but the Inveſtigation of it too. He adds that, Pag. 11. "A common Man would think, "that when he had been taught to find "the Sum of each of theſe Quantities, he "ſhould not want a Series to help him to "add them together."

All the beſt Mathematicians, that have lived within this laſt Century, have been of a different Opinion: all of them have been employed upon ſimilar Subjects; few of them but have been employed upon this very Subject. To mention one, inſtead of many; this Problem, which he deſpiſes ſo much, is in *De Moivre's Miſcellanea Analytica.* —The *Obſerver* can find no Fault with my Solution, and yet he ſeems to imagine, that I myſelf was afraid to truſt it; and therefore, "when all imaginable Equations "lay open before me, I choſe for my Example that one, which was already computed for me in the *Arithmetica Univerſalis.*" (p. 11.) All the Anſwer I can make to this groundleſs Aſſertion is, that, if he will do me the Honour to give me any Equation, (not made difficult on Purpoſe) I will engage to write down the Anſwer, as faſt as I can write with any Care and Attention.

The *Obſerver* goes on to my next Corollary, before he has finiſh'd his Obſervations upon this. "In the next Corollary, it is "propos'd to find the Aggregate of the Sum "of the Roots, of their Squares, Cubes, *&c.* "*in infinitum.* It ſhould have been added, if

" each Root of the Equation be leſs than " 1, or greater than — 1. For within theſe " narrow Limits muſt the Problem be con- " fin'd, or it does not admit an Anſwer."

I anſwer firſt, that what the *Obſerver* ſays of the Limits of this Problem, is true; but that moſt Readers would have thought me a Trifler, if I had mention'd them. If a Writer undertakes to find the Sum of a Series $a + a^2 + a^3 + a^4$ *&c. ad infinitum*, every Reader will underſtand that ($a$) muſt be leſs than (1).

I anſwer 2*dly*, That it is not once in a thouſand Times, that Limitations of this Nature, are mention'd by the beſt Writers.

And laſtly, I anſwer, that this Problem is in *De Moivre*, who does not mention this Limitation, which, in the *Obſerver's* Opinion, was ſo abſolutely neceſſary: and that the ſame Problem is ſolved by Mr. *Stirling*, who forgets too this eſſential Limitation. Indeed, they knew very well, that he who underſtood the Problem, would underſtand the Limitation too; and that the Reader, who ſtood in need of being put in mind of the Limitation, would never underſtand the Problem.

The *Obſerver* however thinks it neceſſary to explain to his Readers, what Abſurdities would follow, if this Limitation was to be neglected. " Suppoſe, for Example, that " one of the Roots of the Equation ſhould be " 4; then we ſhall find by the *Rule before us*, "that $1+4+16+64$, *&c.* is equal to $\frac{1}{-3}$." He has ſaid before, that my Rule is to find the Sum of the Roots, of their Squares, *&c. ad infinitum:* which being true, it is very plain, that the Sum of the Series $1+4+16+256$, *&c.* was not found by my Rule: for (1) is neither (4), nor the Square of (4), nor any other Power of (4). Then again; My Rule requires the *Equation* to be given: but the *Obſerver* finds this Series, without ſuppoſing any *Equation* at all: another Proof that he did not derive his Series from the *Rule before him.* I can eaſily explain this Difficulty. In the Solution of my Problem, I have four Steps of this Form $\frac{1}{1-A} = 1 + A + A^2 + A^3$ *&c.* Subſtitute (4) for *(A)*, and you will have $\frac{1}{-3} = 1 + 4 + 16 + 64$, *&c.* and this is his Series.—With ſo little Attention did the *Obſerver* read the Book, he was writing againſt, that it is very plain, he has here miſtaken theſe Steps in the Inveſtigation of my Rule, for the Rule itſelf.

But he has not yet done with theſe Limits: He goes on,

"But our Calculator will ſay, that he is "not anſwerable for this Miſtake, that he "proceeds according to the famous Binomial "Theorem. As far as he underſtands it, I "ſuppoſe." Indeed I ſhall ſay no ſuch Thing: for the Binomial Theorem never came into my Head, whilſt I was ſolving this Problem; and for a long Time I could not imagine, how the *Obſerver* himſelf came to think of it upon this Occaſion; till at laſt I recollected, that it is in his Diſcourſe upon the Binomial Theorem, where *Saunderſon* teaches this Doctrine of Limits, and how to ſum the Series

$$1+A+A^2+A^3 \text{ \&c, } \textit{ad infinitum.}$$

In the next Page he returns again to my firſt Corollary. " Let us turn back to the " Example in the fifth Page, and we ſhall ſee " how he applies it," (viz. *the Binomial Theorem, which was not at all in my mind, when I wrote this)* " in another Inſtance. There " we have $p=1$, $n=4$. Therefore ſays he " $\frac{p-p^{n+1}}{1-p}=4$. A Man muſt be a deep Mathe" matician indeed, who can diſcover, that $\frac{0}{0}=4$."

As this, of all the *Obſerver*'s Remarks has been moſt talked of, and I am afraid to my Prejudice; I ſhall examine it very fully. But before I enter upon this Examination, I muſt take Notice, how the *Obſerver* has exaggerated my Expreſſion. A Man ſays He *muſt be a deep Mathematician indeed, who can diſcover that* $\frac{0}{0}=4$. Now this is the Examiner's Notation and not mine. I do not even uſe the mark of Equality. I don't write $\frac{p-p^{n+1}}{1-p}=4$; but thus $\frac{p-p^{n+1}}{1-p}$ (4): which I doubted not would be read as I intended it; *viz.* that when $p=1$ and $n=4$, the Expreſſion $\frac{p-p^{n+1}}{1-p}$ becomes (4).

This being premiſed, I go on to ſupport what I have there ſaid, by unqueſtionable Authorities, and I ſhall do it firſt by the Authority of *Mac Laurin.* See Art. 864. p. 698. of his Fluxions.

—— *When any Quantity* (N) *is expreſſed by a Fraction* $\frac{P}{Q}$, *if* P *and* Q *vaniſh at the ſame Time, we are not thence to couclude that* $N=0$. *Thus ſuppoſe* $N=\frac{aa-ax}{a-\sqrt{ax}}$; *and when* $x=a$ *the Numerator and Denominator of* N *vaniſh together:* that is, when $x=a$, then $ax=aa$, and $\sqrt{ax}=a$, and $N=\frac{aa-ax}{a-\sqrt{ax}}=\frac{aa-aa}{a-a}=\frac{0}{0}$; to uſe the candid Notation of the *Obſerver.* Mac Laurin goes on. *But if we reduce the Value* N *to a more ſimple Form, by dividing the Numerator and Denominator by their common Diviſor* $\sqrt{a}-\sqrt{x}$; *we ſhall find* $N=a\times\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}}=$ (*when* $x=a$) $=a\times\frac{2\sqrt{a}}{\sqrt{a}}=2a$*:* That is, if we interpret *MacLaurin,* as the *Obſerver* interprets Me, $\frac{0}{0}=2a$*:* for $N=\frac{aa-ax}{a-\sqrt{ax}}=\frac{aa-aa}{a-a}=\frac{0}{0}$ when $x=a$.

But $N = 2a$, therefore $\frac{0}{0} = 2a$.
Or make $a = 2$; then $2a = 4$ and $\frac{0}{0} = 4$.
Now I will apply what *Mac Laurin* ſays, upon this Occaſion, to my own Fraction; which is $\frac{p - p^{n+1}}{1 - p}$, or when $n = 4$; which is the Caſe under Debate, $\frac{p - p^5}{1 - p}$.

Let therefore $N = \frac{p - p^5}{1 - p}$: Now if $(p = 1)$ both the Numerator and Denominator vaniſh together; for $p - p^5 = 1 - 1 = 0$. And $1 - p = 1 - 1 = 0$. But *Mac Laurin* ſays, that we are not to conclude from thence, that the Value of the Fraction, or $N = 0$. But we muſt reduce it to a more ſimple Form, and take the Value, which that more ſimple Form gives us, for the Value of the Fraction. Now the *Obſerver* himſelf, had reduced this Fraction to a more ſimple Form: how he reduced it ſhall be taken Notice of by and by: but he had himſelf reduced it to this Form $p + p^2 + p^3 + p^4 = 4$, when $p = 1$.

I may conclude from hence upon *Mac-Laurin*'s Authority;

1*ſt*, That though both the Numerator and Denominator of a Fraction, by a particular Subſtitution, become (0); yet it does not from thence follow, that the Value of the Fraction itſelf vaniſhes at the ſame Time.

2*dly*, That the *Obſerver* himſelf had proved the Value of my Fraction to be (4), by reducing it to a more ſimple Expreſſion; but did not know that his Proof was a good one. But to proceed,

The *Obſerver* in the Beginning of his Obſervations has quoted *Saunderſon*'s Diſcourſe on the Reſolution of Biquadratic Equations *by the Mediation of Cubics*: (Vol. 2, p. 735, Art. 466, *&c.* of his Algebra,) and all that he ſays to the purpoſe upon my firſt Lemma, is taken from thence.

That Expreſſion *by the Mediation of Cubics*, to the Meaning of which he did not ſufficiently attend, led him into the Miſtake which I have taken Notice of before; where he ſuppoſes, that a Cubic alone is ſufficient for this Reduction, becauſe its three Roots, by *only* changing the Signs, may have each two Values: (*i. e.* ) may become *ſix* Roots. I wiſh he had read this Diſcourſe quite through, ſince it does not take up above five Pages; and as the Title to it had drawn him into the Error, which I have juſt mentioned; ſo the laſt Article of it would have prevented his falling into the Miſtake, which I am now examining. For in this Article, (469) the Profeſſor falls upon a Fraction, whoſe Numerator ($r$) is equal to (o) and Denominator ($e$) equal to (o) alſo: notwithſtanding which he ſets himſelf to find the Value of the Fraction $\frac{r}{e}$, and does find that $\frac{r}{e} = \frac{3}{1} = 3$: or, in the Notation of the *Obſerver*, $\frac{r}{e} = \frac{0}{0} = \frac{3}{1} = 3$.

He ſays, in the next Page, that in every other Expreſſion the Difficulties are *doubled*. I ſhall ſhew therefore how they may be all reduc'd to a ſimpler Form, and the Value of them found by *Mac Laurin*'s Rule.

And firſt for this $\frac{p-p^5}{1-p}$. The *Obſerver* himſelf has reduc'd this as follows,

$$\frac{p}{1-p}=p+p^2+p^3+p^4+p^5+p^6 \;\&c.$$

$$\frac{p^5}{1-p}=\times \quad \times \quad \times \quad \times \quad p^5+p^6 \;\&c.$$

therefore

$$\frac{p-p^5}{1-p}=p+p^2+p^3+p^4.$$

Thus by the Help of the *Binomial Theorem*, and *two infinite* Diviſions, he has found out, that $\frac{p-p^5}{1-p}=p+p^2+p^3+p^4$: Which indeed is nothing but the Quotient of the eaſieſt Sum in common Diviſion that can well be imagin'd.

Here is the Operation:

$$\begin{array}{l}
1-p\,)\;p \times \times \times -p^5\;(p+p^2+p^3+p^4 \\
\quad\quad\;\underline{p-p^2} \\
\quad\quad\quad +p^2 \\
\quad\quad\quad \underline{+p^2-p^3} \\
\quad\quad\quad\quad\quad +p^3 \\
\quad\quad\quad\quad\quad \underline{+p^3-p^4} \\
\quad\quad\quad\quad\quad\quad\quad +p^4-p^5 \\
\quad\quad\quad\quad\quad\quad\quad \underline{+p^4-p^5} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad \bullet
\end{array}$$

What I have ſaid of this, is true of all the reſt: they are all reducible without the extraordinary Aid of infinite Series. Take any one of them, and divide the Numerator by the Denominator, and the Quotient will be the value of the Fraction, if you make $p=1$.

I have therefore the Authority of both *Mac Laurin* and *Saunderſon*, to ſay, that the Value of a Fraction may be *real*, though both the Numerator and Denominator be each of them (0). And I have the Authority of *Mac Laurin*'s Rule, for giving the Fractions, in my Example, thoſe particular Values, which I have given them.

If even this ſhould not be thought enough, I have other Sponſors, who will anſwer both for the Problem and Example together. The *Obſerver* may conſult for this Purpoſe that *excellent Analyſt De Moivre*, whom he mentions ſo often; and in pag. 165,&c. he will diſcover, how much I have been obliged to *Nicholas Bernoulli*, *Mommort*, and *De Moivre*, in finding my Series; and that I have the Authority of *Nicholas Bernoulli* and *De Moivre*, for the Application of it, to this particular Caſe, where $p=1$.

The Authority of ſuch Writers cannot fail to ſupport me; but indeed, if the Prejudice had not been againſt me, I had wanted no Authority at all. What the *Obſerver* repreſents, as ſuch a Depth of Myſtery, is no Myſtery; but the plaineſt and ſimpleſt

Thing that can be imagined. I did not find the Value of $\frac{p-p^5}{1-p}$ by infinite Series, as the *Obſerver* does; nor yet by Mr. *Mac Laurin*'s Rule; nor even by common Diviſion. But I knew that it ſtood for $p+p^2+p^3+p^4$; which muſt be equal to (4), when $p=1$. This Expreſſion $\frac{p-p^{n+1}}{1-p}$ (I will not ſay is equal to) but ſtands for $p+p^2+p^3+p^4 \dots p^n$. Now when $p=1$, then $p+p^2+p^3+p^4 \dots p^n = n$, the Number of Terms. In this caſe therefore $\frac{p-p^{n+1}}{1-p}$ ſtands for ($n$). But why may I not ſay that $\frac{p-p^{n+1}}{1-p}$ is equal to $p+p^2+p^3+p^4 \dots p^n$. always? Since this Propoſition is in almoſt every Treatiſe of Algebra that is extant, and none of them, that I remember, except the Caſe of $p=1$.

“ But, if our Mathematician will give “ me Leave, I will ſet him free at “ once from this Kind of Embarraſment: “ It ſhall never diſturb his Calculation a-“ gain. Let him obſerve then, that when “ ($p$) is 1, theſe Diviſions do juſt no-“ thing at all. Continue them as long as “ you pleaſe, and the Remainder is always “ the ſame with the Quantity which was firſt “ to be divided. If ($p$) is more than 1, “ then the Remainder continually increaſes, “ and therefore the farther you continue “ the Series, the more diſtant is the Quo-“ tient from the Truth.”

To this I anſwer firſt, that what the *Obſerver* ſays here, can be no Diſcovery to his *Cambridge* Readers, for whom alone he intended his Obſervations. All of us in this Place read *Saunderſon*, from whoſe Diſcourſe on the Binomial Theorem all this is taken. The worthy Profeſſor introduces it with a little Parade: He ſays it might puzzle *even an ingenious Reader*, but he adds, *not very converſant in theſe Affairs.*

I anſwer 2*dly*, That however uſeful this Caution might be to the *Obſerver* himſelf, who found out, by two infinite Diviſions, that $\frac{p-p^5}{1-p} = p + p^2 + p^3 + p^4$:

It certainly could have been of no Service to me; ſince I found out the ſame Thing, by one plain finite Diviſion; or rather, which is the Fact, knew it to be ſo, without ſeeking.

I anſwer 3*dly*, That it has been of no Advantage to the *Obſerver* himſelf, but rather a Prejudice to him: ſince it has made him conclude, that *(p)* muſt be leſs than $(+1)$, or greater than $(-1)$, which is a Miſtake.

He began with the Suppoſition of $p$ leſs than $(1)$; but he might have ended with $p$ greater than $(1)$. For if $\frac{p-p^5}{1-p} = p + p^2 + p^3 + p^4$; it is certain, that $\overline{1-p}$ is a *Diviſor* of $\overline{p-p^5}$; and that $\overline{p+p^2+p^3+p^4}$ is the Quotient. But if $\overline{1-p}$ is a *Diviſo* of $\overline{p-p^5}$, on the Suppoſition of $p$ leſs than

(1); it is an indiſputable Concluſion, that $\overline{1-p}$ will alſo *divide* $\overline{p-p^5}$; though ($p$) ſhould be greater than (1): and that $p+p^2+p^3+p^4$ will be ſtill the Quotient.

The *Obſerver* goes on, " there is, I think, " ſomewhere in Prof. *Saunderſon's* Algebra, " an Inſtance or two of this kind; with which " he uſed, *en badinant*, to entangle his young " Scholars. And he might well excuſe *them*, if " they were not able to unravel the Knot. " For he knew that the Doctrine of Infinites, which is of ſuch extenſive Uſe, and " incomparable Elegance, under the management of a Maſter; often betrays Men of " lower Abilities into inextricable Difficulties." My Anſwer is, that whatever the worthy Profeſſor may have ſaid of *Infinity*, on other Occaſions; he moſt certainly would not have ſaid one word about it on this Occaſion. We have nothing to do here with Infinity. It is a matter of plain common Diviſion of Numerator by Denominator.

I was ſolicitous to anſwer this Part of the *Obſerver*'s Pamphlet minutely, but I ſhall be ſhorter upon the reſt of it.

Lemma III, I hope, is not ſo obſcure as the *Obſerver* repreſents it. He himſelf will, I believe, be thought ſo by his Readers on this Subject. But if I underſtand him right, the *Exceptions*, which he wanted, are mentioned in my Lemma. However he does not ſay my Lemma is falſe, or that I have miſapplied it. So I proceed to the four Problems.

Theſe four Problems he ſays, contain *whatever has any merit in this Chapter.* But I hope few of my Readers will be of his Opinion; and, to ſay nothing of my *firſt* and *third* Lemma, the Authorities, which I have cited, to the Credit of my *ſecond* and its *Corollaries,* muſt I think have given him a better Opinion of it.

In his Animadverſions upon the firſt and ſecond Problems, he proceeds upon an Error of the Preſs, which a little Attention might have diſcovered. The Problem is propoſed generally: my Solution of it was divided into two Articles. 1ſt, I ſolved the ſpecial Caſe of the Square, which he acknowledges to be *neat*; and 2dly, the general Caſe. The Printer has left (1), the mark of the 1ſt Article; but has made the 2d. Article into a new Problem.

The general ſolution, he ſays, is given by a *Series, of which every Term conſiſts of other Serieſes.* He would have his Reader underſtand *infinite Serieſes.* For unleſs that be his meaning, all that he ſays comes to no more than this; that I uſe *ſingle* Letters to expreſs compound Quantities, which is the Fact. And all Writers do the ſame.

What he ſays of the great Things that have been done by *Des Cartes* and his Commentators, in the matter of Subſtitutions; if it proves any thing, proves as much againſt all the Writers of a later Age, as it proves againſt me; which I hope will be thought a ſufficient Anſwer.

Concerning my Subſtitutions, which he ſuppoſes to be new, he ſays (p.20) "they" (meaning *Des Cartes* and his Commentators) "thus transformed Equations, in order to "reduce them, either to lower Dimenſions, "or to fewer Terms. And they neglected "other Transformations, becauſe it did "not appear to them, nor has our Author "attempted to ſhew it, that they could be "of any additional advantage in ſearching "for the Roots of Equations." — I anſwer, that I have attempted to ſhew it in my fifth Chapter.

As the *Obſerver* came freſh from the Reading of *Des Cartes* and his Commentators, what he ſays of my third Problem is not to be accounted for, as the Solution of it is derived from their plain, ſimple Subſtitutions. His words are, "The Solution of the third Problem is right, whenever (*n*) is an even Number. We ſhould "have been told, how to change it, when "(*n*) is odd." My anſwer is, that it need not be changed at all. Even and odd have nothing to do in the Caſe; as will appear by the following Example, where (*n*) is odd.

The Roots of the Cubic Equation $x^3-6x^2+11x-6=0$; are 1, 2, 3. Now my Problem is to find an Equation, whoſe Roots ſhall be $1+2$, $1+3$, $2+3$. In order
or 3, 4, 5.

to do this

$$\begin{aligned}
\text{I make}\quad x &= 6-u \quad\text{then}\\
x^3 &= 216-108u+18u^2-u^3\\
-x^2 &= -216+72u-6u^2\\
+11x &= 66-11u\\
-6 &= -6\\
\text{and conſequently}\quad 0 &= 60-47u+12u^2-u^3.
\end{aligned}$$

Now *Des Cartes*, and all the Writers of Algebra, prove, that the Roots of the new Equation $0=60-47u+12u^2-u^3$, are $6-1, 6-2, 6-3$, or $5, 4, 3$.

The new Equation therefore has the Roots that were required in the Problem. And the Reader, who underſtands any thing of theſe Matters, will be puzzled to find out the *Obſerver*'s Difficulty. I believe I have diſcovered it. The *Obſerver* has been accuſtomed to ſee the higheſt Terms of Equations *poſitive:* and in this Example the higheſt Term is *negative* ($-u^3$). And he thinks, that the Operation is not compleat, till we have changed all the Signs, and made it

$$u^3-12u^2+47u-60=0.$$

And this is the great *Change*, of which, it ſeems, I ought to have advertiſed my Reader: as if it was not a Matter of Faſhion, rather than Neceſſity, to make the higheſt Term of the Equation *poſitive.* Nay, if he will conſult Mr. *Maſeres*'s Diſſertation on the Uſe of the *Negative Sign*; He will find, that this Faſhion often leads us into a Sort of Impropriety, which that careful Writer ſtudiouſly avoids.

I proceed now to his laſt Objection. "Let us next ſee what Uſe he makes of "Symbols. My Inſtance of his want of At-"tention, ſhall be taken from a Place, "where the Operations are too plain to "leave Room for any Diſpute; and the "Miſtakes too often repeated, to be im-"puted to Errors of the Preſs. In the "51ſt Page, the Writer has Occaſion to di-"vide one Quantity by another, and to "find the Remainder: and ſo much does he "deſpiſe his Readers, that he blunders in "almoſt every ſtep of this eaſy Proceſs. I "need not tranſcribe it, in order to point "out the Miſtakes. There is (+) inſtead of "(—) in the Quotient; and the Remainder "ſhould have been $9x^4 + 12x^2 + 2x + 6$: "which agrees with the Remainder found "by our Analyſt in only ONE TERM."

ANSWER.

This Diviſion conſiſts of two Operations. In the firſt Operation there is an Error of the Preſs: the Vinculum is omitted more than once. I am very ſorry for it; but there is not a ſingle Blunder in the Operation.

In the next Operation he ſays truly, that there is (+) inſtead of (—) in the Quotient. He might have ſaid too, that in Conſequence of this, there is (+) inſtead of (—), in the Remainder alſo. Theſe I confeſs are my own Blunders, ſince he will needs call them by ſo hard a Name. But he adds, that my Remainder agrees with his in only one

TERM. I anſwer, that my Remainder agrees with his in EVERY TERM: and that the only Error, I have committed, conſiſts in the Omiſſion of the Sign (—), which I have acknowledged before.

My Remainder is,

$$4x^{2}+5x+6-\overline{3xx+x}\times\overline{3xx+x+3}.$$

It ſhould be,

$$4x^{2}+5x+6-\overline{-3xx+x}\times\overline{3xx+x+3}.$$

The Reader will obſerve, that ($3xx$) under the firſt Vinculum, in the firſt Line, has no Sign before it, and therefore is to be underſtood (+): whereas it ought to have had the Sign (—) before it, as it has in the ſecond Line; and this is all the Error.

This then I have confeſſed from the Beginning, that the *Obſerver* had found one Miſtake in my Book. He found *one* (+) inſtead of a (—). And if he had looked carefully he might have found, in the ſame Place, *another*.

An Error of this Nature might have eſcaped the moſt careful Writer: and for that Reaſon, I hope, needs no Apology. And yet ſome Sort of Apology might be made for it. It is in a Place, where even Careleſſneſs might be pardonable; if indeed it can ever be ſo. The Diviſion in which he found this (+) inſtead of a(—), is ſet down only to ſhew the Courſe of a Proceſs. I make no Uſe either of the Quotient, or of the Remainder. A Reader, who read only for Information, would ſee, at the firſt Glance of his Eye, all that I wanted to ſhew him: and would never give himſelf the Trouble to examine a ſingle Step in the Operation. As no Concluſion was intended to be drawn from this Remainder, I did not work it out, as the *Obſerver* has done, into ſimple Terms. But, what any other Writer would have done, in a like Caſe, I indicated, by a Notation, how the ſimple Terms might be found; little imagining that any Reader would ever take the Pains to find them. My Notation is not very common: I wiſh I had uſed the moſt common Notation, ſince it might have prevented the *Obſerver* from falling into this Miſtake. But my Notation, though not the moſt common, is not peculiar to me. And if it were ſo, I am ſure it is ſufficiently analogous to the common Notation, to be very intelligible to a Reader of any Attention, and Experience in theſe Studies. Indeed, in a plain common Diviſion, ſuch as this is, the moſt ſingular Notation could not perplex a Mathematician; as the Simplicity of the Proceſs would of itſelf plainly interpret the Notation.

And thus I have anſwered every particular Objection, that the *Obſerver* has made to my Book. As to his many, cruel, general Accuſations, I have taken no Notice of them.

8 JY 62

The good natured Reader will conſider the Anxiety of my preſent Situation, and will make ſome Allowance too for the Haſte, in which this Reply has been written. Many more Miſtakes may have eſcaped me in this ſhort Pamphlet, than the *Obſerver* has found in my Book: I ſubmit them to the Candour of my Readers.

*Magd. Coll. Camb.*
*Jan.* 25. 1760.

*THE END.*

8 JY 62